Nonsmooth Implicit Differentiation for Machine Learning

Jérôme Bolte, Tam Le, Edouard Pauwels, and Antonio Silveti-Falls




Neurips Poster (2021)

Outline

1. Smooth implicit function theorem.
2. Nonsmooth implicit function theorem of Clarke.
3. Path differentiable nonsmooth implicit function theorem (with calculus).
4. Applications
5. What can go wrong?

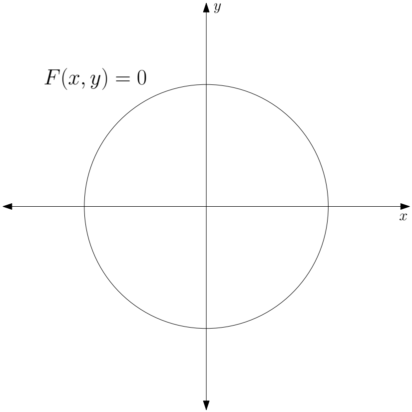
Implicit Functions

Consider the smooth function

$$F(x, y) = x^2 + y^2 - 1$$

and the equation

$$F(x, y) = 0.$$



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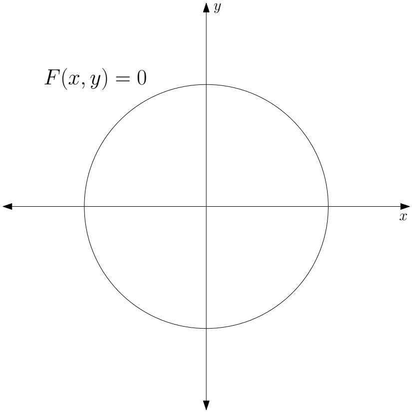

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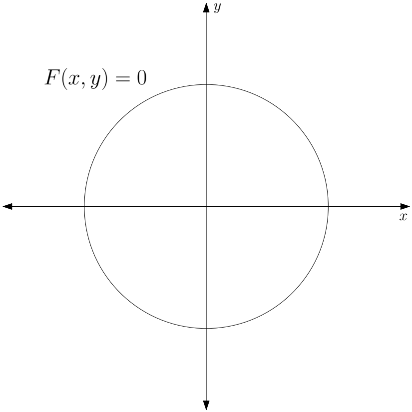

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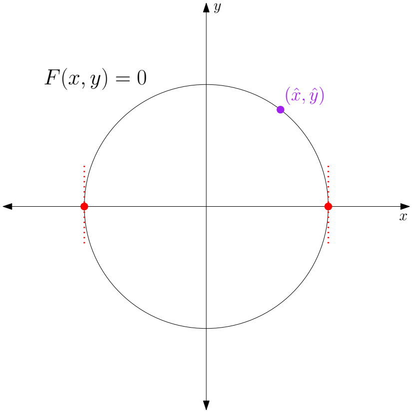

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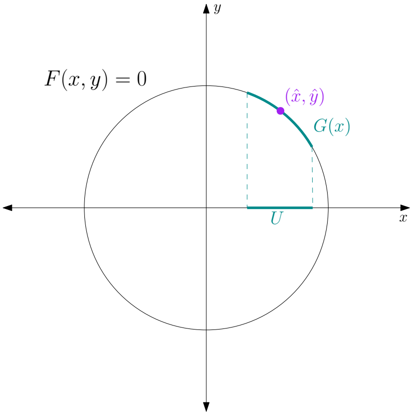

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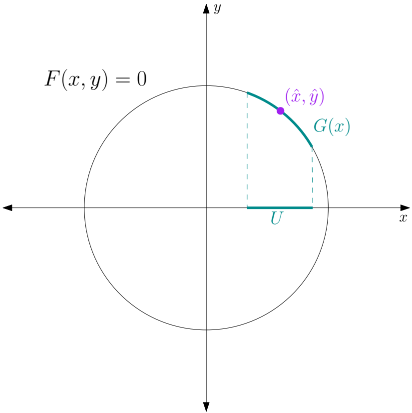

Locally Lipschitz Functions and the Clarke Subdifferential

Definition (Locally Lipschitz-continuous)

A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz-continuous if, $\forall x \in \mathbb{R}^n$, $\exists$ a neighborhood $U \subset \mathbb{R}^n$ of x and $c > 0$ such that, $\forall y, z \in U$,

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Definition (Clarke subdifferential (1983))

Given a locally Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}^m$, the Clarke subdifferential at a point $x \in \mathbb{R}^n$ is

$$\partial^c F(x) = \mathrm{conv}\left(\left\{\lim_{k\to\infty} J_F(x_k) : x_k \in \mathrm{diff}_F \text{ and } x_k \to x\right\}\right).$$

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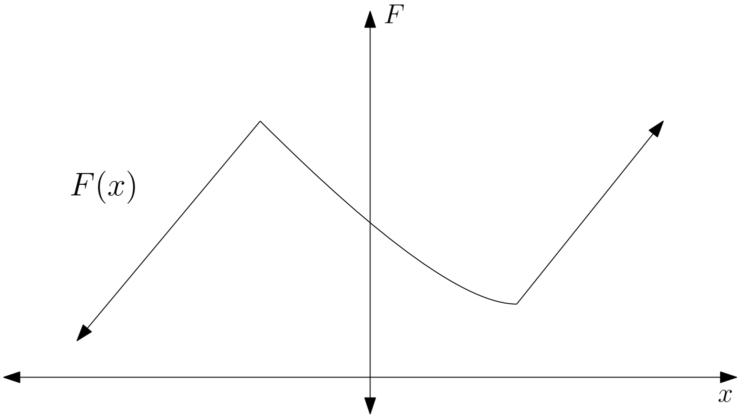

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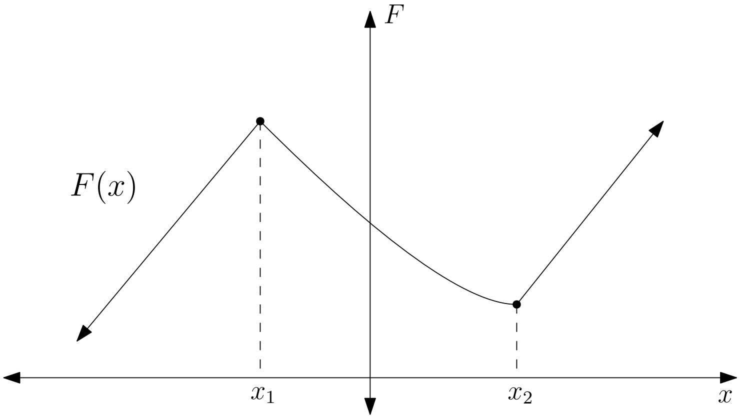

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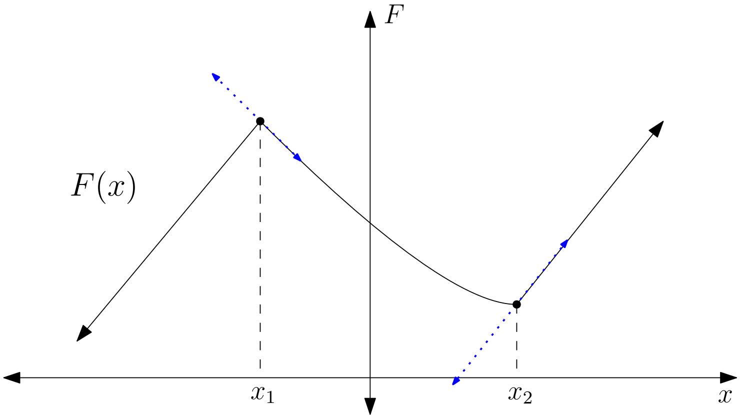

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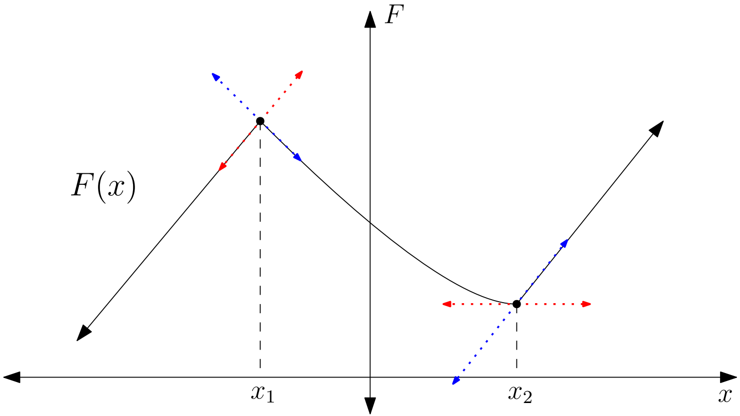

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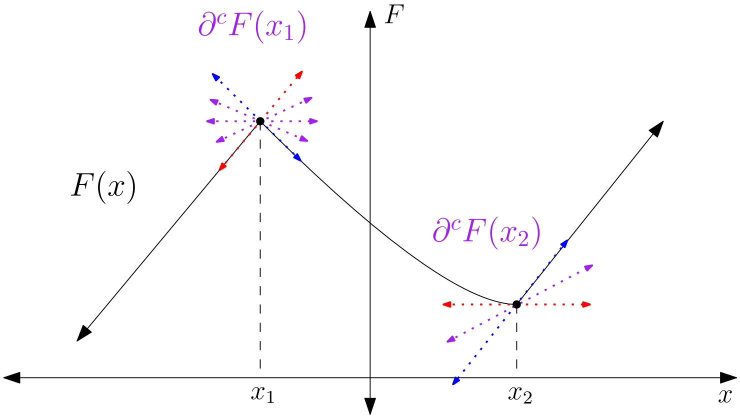

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Nonsmooth Implicit Function Theorem of Clarke

Theorem (Clarke 1983)

Let $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ *be locally Lipschitz and* $(\hat{x}, \hat{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ *such that*

$$F(\hat{x}, \hat{y}) = 0.$$

If, $\forall [A\ B] \in \partial^c F(\hat{x}, \hat{y})$, B *is invertible, then* $\exists U \subset \mathbb{R}^n$ *a neighborhood of* $\hat{x}$ *and a locally Lipschitz function* $G(x)$ *so that*

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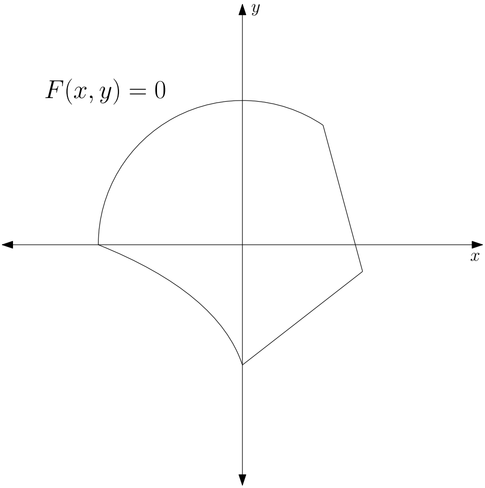

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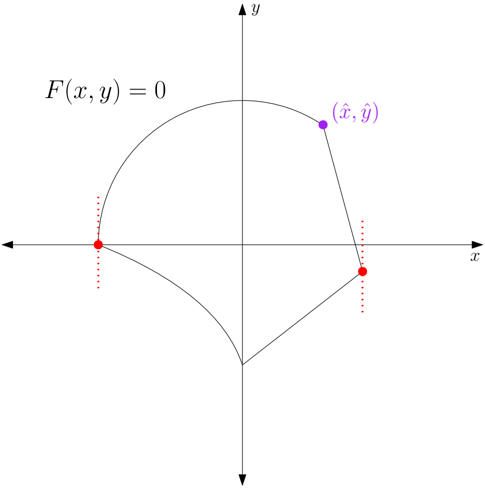

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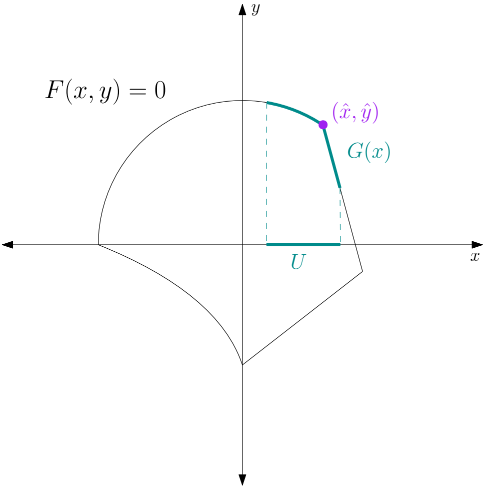

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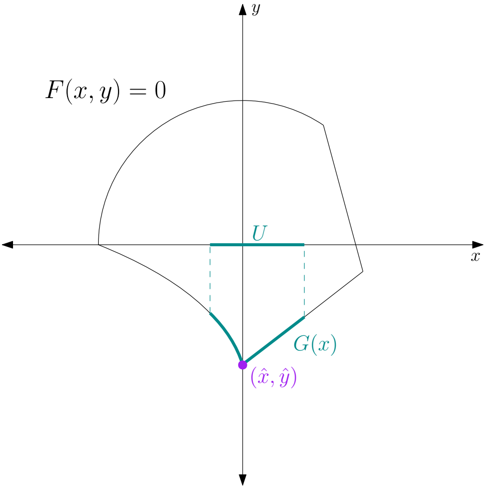

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No - need something beyond ∂^c.

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A set valued mapping $D_F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a conservative field (or conservative Jacobian) for $F : \mathbb{R}^n \to \mathbb{R}$ locally Lipschitz if:

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3. For any absolutely continuous curve $\gamma : [0,1] \to \mathbb{R}^n$,

$$\frac{d}{dt} F(\gamma(t)) = \langle u, \dot{\gamma}(t) \rangle \qquad \forall u \in D_F(\gamma(t))$$

   for almost all $t \in [0,1]$.

We call F path differentiable.

Path Differentiable Nonsmooth Implicit Function Theorem

Theorem

Let $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ *be path differentiable and* $(\hat{x}, \hat{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ *be such that*

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The conservative Jacobian of G is given by

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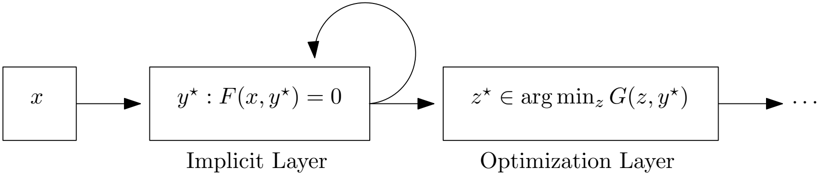
Deep Learning with Implicit Layers



- Deep equilibrium networks [Shaojie Bai, J. Zico Kolter, Vladlen Koltun 2019], Monotone deep equlibirium networks [Ezra Winston, J. Zico Kolter 2020].
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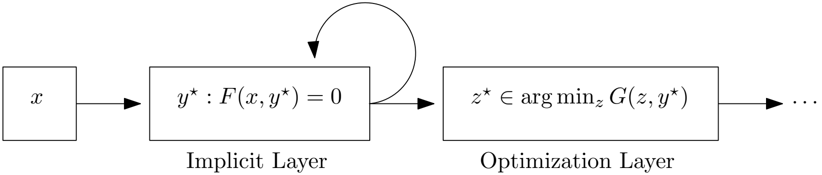

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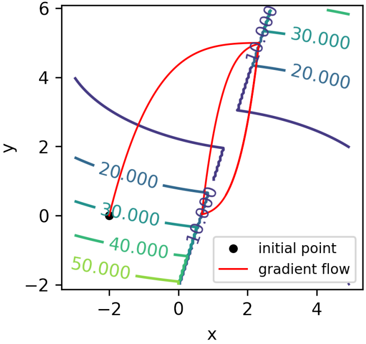
- Convergence guarantees for training.

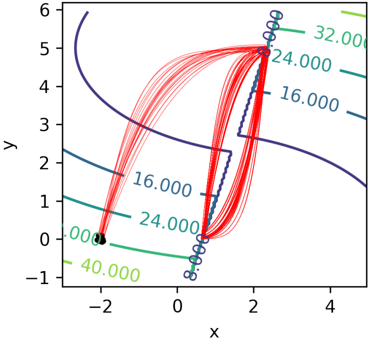
Pathological Examples - Cycles

Gradient descent type algorithm (using backprop) applied to:

$$\min_{x,y,s} \quad \ell(x,y,s) \stackrel{\text{def}}{=} (x - s_1)^2 + 4(y - s_2)^2$$

$$\text{s.t.} \quad s \in \arg\max\{(a+b)(-2x+y+2) : a \in [0,3], b \in [0,5]\}.$$





Pathological Examples - Lorenz Attractor

For $u \in \mathbb{R}^3$, define $L(u) \stackrel{\text{def}}{=} \left(10(y-x), x(28-z)-y, xy-\frac{8}{3}z\right)$.

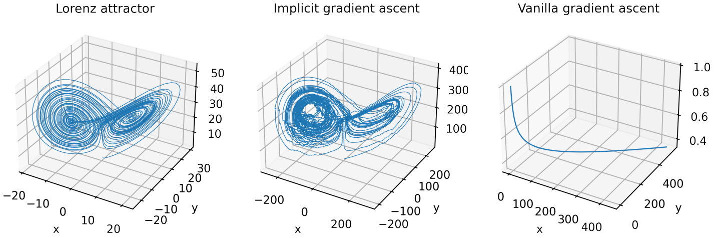
Implicit formulation

$$\max_{u \in \mathbb{R}^3} u^T z \quad \text{s.t.}$$

$$z \in \operatorname*{argmin}_{s \in \mathbb{R}^3} \|s - L(u)\|^4$$

Explicit (vanilla) formulation

$$\max_{u \in \mathbb{R}^3} u^T L(u)$$



Thanks for Listening

Thanks for listening.

Full paper available on arxiv: https://arxiv.org/abs/ 2106.04350

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